

MINIMAL LAGRANGIAN CONNECTIONS ON COMPACT SURFACES

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ABSTRACT. We classify the torsion-free minimal Lagrangian connections on the tangent bundle of a compact oriented surface of non-vanishing Euler characteristic.

1. INTRODUCTION

A *projective surface* is a pair (Σ, \mathfrak{p}) consisting of a smooth surface Σ and a *projective structure* \mathfrak{p} , that is, an equivalence class of torsion-free connections on the tangent bundle $T\Sigma$, where two such connections are called *projectively equivalent* if they share the same geodesics up to parametrisation. A projective surface (Σ, \mathfrak{p}) is called *properly convex* if it arises as a quotient of a properly convex open set $\tilde{\Sigma} \subset \mathbb{RP}^2$ by a free and cocompact action of a group $\Gamma \subset \mathrm{SL}(3, \mathbb{R})$ of projective transformations. The geodesics of \mathfrak{p} are the projections to $\Sigma = \Gamma \backslash \tilde{\Sigma}$ of the projective lines \mathbb{RP}^1 contained in $\tilde{\Sigma}$. In particular, locally the geodesics of a properly convex projective structure \mathfrak{p} can be mapped diffeomorphically to straight lines, that is, \mathfrak{p} is *flat*.

Properly convex projective surfaces are of particular interest as they may be seen – through the work of Hitchin [8] and Choi & Goldman [4] – as the natural generalisation of the notion of a hyperbolic Riemann surface. Labourie [9] and Loftin [11] have independently shown that on a compact surface of negative Euler characteristic there exists a one-to-one correspondence between properly convex projective structures and pairs (J, C) consisting of a complex structure J and a cubic differential C that is holomorphic with respect to J . Since then Benoist & Hulin [2] have extended this result to noncompact projective surfaces with finite Finsler volume and Dumas & Wolf [5] study the case of polynomial cubic differentials on the complex plane.

In another direction it is shown in [6] that a projective structure \mathfrak{p} on a surface Σ canonically defines a split-signature anti-self-dual Einstein metric $h_{\mathfrak{p}}$, as well as a symplectic form $\Omega_{\mathfrak{p}}$ on the total space of the affine bundle $A \rightarrow \Sigma$, whose underlying vector bundle is the cotangent bundle of Σ . Moreover, the \mathfrak{p} -representative connections turn out to be in one-to-one correspondence with the sections of $A \rightarrow \Sigma$. Therefore, fixing a representative connection $\nabla \in \mathfrak{p}$ gives a section $s_{\nabla} : \Sigma \rightarrow A$ and hence an isomorphism

$\psi_\nabla : T^*\Sigma \rightarrow A$, by declaring the origin of the affine fibre A_p to be $s_\nabla(p)$ for all $p \in \Sigma$. Correspondingly, we obtain a pair $(h_\nabla, \Omega_\nabla) = \psi_\nabla^*(h_{\mathfrak{p}}, \Omega_{\mathfrak{p}})$ on the total space of the cotangent bundle. Besides being a geometric structure of interest in itself (see [6] for details), the pair $(h_\nabla, \Omega_\nabla)$ has the natural property

$$o^*h_\nabla = (s_\nabla)^*h_{\mathfrak{p}} = -\text{Ric}^+(\nabla) \quad \text{and} \quad o^*\Omega_\nabla = (s_\nabla)^*\Omega_{\mathfrak{p}} = \frac{1}{3}\text{Ric}^-(\nabla),$$

where $o : \Sigma \rightarrow T^*\Sigma$ denotes the zero-section and $\text{Ric}^\pm(\nabla)$ the symmetric – and anti-symmetric part of the Ricci curvature $\text{Ric}(\nabla)$ of ∇ . Consequently, we call ∇ *Lagrangian* if the Ricci tensor of ∇ is symmetric, or equivalently, if the zero-section o is a Lagrangian submanifold of $(T^*\Sigma, \Omega_\nabla)$. Likewise, we call ∇ *timelike/spacelike* if $\pm\text{Ric}^+(\nabla)$ is positive definite, or equivalently, if the zero-section o is a timelike/spacelike submanifold of $(T^*\Sigma, h_\nabla)$. Here and henceforth, the upper signs correspond to the timelike case and lower signs to the spacelike case. Moreover, we call ∇ *minimal* if the zero-section is a minimal submanifold of $(T^*\Sigma, h_\nabla)$.

Using the results of Labourie [9], the author proved in [12] that a properly convex projective structure \mathfrak{p} on a compact oriented surface with $\chi(\Sigma) < 0$ is defined by a unique representative connection $\nabla \in \mathfrak{p}$, which is spacelike and minimal Lagrangian. Conversely, if ∇ is a spacelike minimal Lagrangian connection on a compact oriented surface Σ with ∇ defining a flat projective structure $\mathfrak{p}(\nabla)$, then (Σ, \mathfrak{p}) is a properly convex projective surface.

Here we show that a minimal Lagrangian connection ∇ on an oriented surface Σ defines a triple (g, β, C) on Σ , consisting of a Riemannian metric g , a 1-form β and a cubic differential C , so that the following equations hold

$$(1.1) \quad K_g = \pm 1 + 2|C|_g^2 + \delta_g\beta, \quad \bar{\partial}C = (\beta - i \star_g \beta) \otimes C, \quad d\beta = 0.$$

As usual, $\bar{\partial}$ denotes the “del-bar” operator with respect to the integrable almost complex structure J induced on Σ by $[g]$ and the orientation, \star_g , δ_g and K_g denote the Hodge-star, co-differential and Gauss curvature with respect to g . Finally, $|C|_g$ denotes the pointwise tensor norm of C with respect to the Hermitian metric induced by g on the third power of the canonical bundle of Σ .

Moreover, we show that a minimal Lagrangian connection defines a flat projective structure if and only if β vanishes identically. Of course, in the projectively flat case the above equations are well-known. The first equation is known as Wang’s equation in the affine sphere literature. In [14], Wang related its solutions to certain affine spheres, see in particular [10] for a nice survey. Moreover, Labourie [9] (see also [1]) interpreted the first two equations as an instance of Hitchin’s Higgs bundle equations [7]. It appears likely that in the case with $\beta \neq 0$ the above triple of equations still admits an interpretation as ‘Higgs bundle equations’ with a non-holomorphic Higgs

field as well as an interpretation in terms of affine differential geometry, but we do not investigate this here.

Note that the last two of the equations (1.1) say that the cubic differential C is *conformally holomorphic*, that is, locally there exists a (real-valued) function r so that $e^{2r}C$ is holomorphic. As a consequence of this one can show that the only examples of minimal Lagrangian connections on the 2-sphere S^2 are Levi-Civita connections of metrics of positive Gauss curvature.

Furthermore, if $(\Sigma, [g])$ is a compact Riemann surface of negative Euler characteristic $\chi(\Sigma)$, then the metric g of the triple (g, β, C) is uniquely determined in terms of $([g], \beta, C)$. Using Hodge decomposition, this is achieved by proving existence and uniqueness of a smooth minimum of the following functional defined on the Sobolev space $W^{1,2}(\Sigma)$

$$\mathcal{E}_{\kappa, \xi} : W^{1,2}(\Sigma) \rightarrow \mathbb{R}, \quad u \mapsto \frac{1}{2} \int_{\Sigma} |du|_{g_0}^2 - 2u - \kappa e^{2u} + \xi e^{-4u} d\mu_{g_0},$$

where $\kappa, \xi \in C^\infty(\Sigma)$ satisfy $\kappa < 0$, $\xi \geq 0$ and g_0 denotes the hyperbolic metric in the conformal equivalence class $[g]$.

An immediate consequence of the first equation in (1.1) and the Gauss–Bonnet theorem is that the area of $o(\Sigma) \subset T^*\Sigma$ with respect to h_∇ satisfies

$$\text{Area}(o(\Sigma)) = \pm 2\pi\chi(\Sigma) + 2\|C\|_g^2.$$

Consequently, we call a minimal Lagrangian connection *area minimising* if the cubic differential C vanishes identically. We conclude by showing that on a compact oriented surface Σ with $\chi(\Sigma) < 0$ there is a one-to-one correspondence between Lagrangian connections that are area minimising and pairs $([g], \beta)$ consisting of a conformal structure and a closed 1-form. It also follows that on such a surface there is a one-to-one correspondence between minimal Lagrangian connections on Σ (that are not area minimising) and pairs $([g], C)$ consisting of a conformal structure and a cubic differential (that does not vanish identically and) that is conformally holomorphic.

Naturally, given our results and the results of Labourie, Loftin, one would expect that on a compact surface Σ of negative Euler characteristic there is a one-to-one correspondence between what one might call ‘*generalised properly convex projective structures*’ and pairs (J, C) consisting of a complex structure J and a cubic differential C that is merely conformally holomorphic with respect to J . This will be investigated elsewhere.

As a by-product, we obtain an identity which may be of independent interest. Denoting by $\mathfrak{P}(\Sigma)$ the space of projective structures on a compact oriented Riemann surface $(\Sigma, [g])$, we show that

$$\sup_{p \in \mathfrak{P}(\Sigma)} \inf_{\nabla \in p} \int_{\Sigma} \text{tr}_g \text{Ric}(\nabla) d\mu_g = 4\pi\chi(\Sigma).$$

Acknowledgements. The author is grateful to Norbert Hungerbühler, Tobias Weth and Luca Galimberti for helpful conversations or correspondence.

2. PRELIMINARIES

Throughout the article Σ will denote an oriented smooth 2-manifold without boundary. All manifolds and maps are assumed to be smooth and we adhere to the convention of summing over repeated indices.

2.1. The coframe bundle. We denote by $v : F \rightarrow \Sigma$ the bundle of orientation preserving coframes whose fibre at $p \in \Sigma$ consists of the linear isomorphisms $f : T_p \Sigma \rightarrow \mathbb{R}^2$ that are orientation preserving with respect to the fixed orientation on Σ and the standard orientation on \mathbb{R}^2 . Recall that $v : F \rightarrow \Sigma$ is a principal right $\mathrm{GL}^+(2, \mathbb{R})$ -bundle with right action defined by the rule $R_a(f) = f \cdot a = a^{-1} \circ f$ for all $a \in \mathrm{GL}^+(2, \mathbb{R})$. The bundle F is equipped with a tautological \mathbb{R}^2 -valued 1-form $\omega = (\omega^i)$ defined by $\omega_f = f \circ v'_f$, and this 1-form satisfies the equivariance property $R_a^* \omega = a^{-1} \omega$. A torsion-free connection ∇ on $T\Sigma$ corresponds to a $\mathfrak{gl}(2, \mathbb{R})$ -valued connection 1-form $\theta = (\theta_j^i)$ on F satisfying the structure equations

$$(2.1) \quad d\omega = -\theta \wedge \omega,$$

$$(2.2) \quad d\theta = -\theta \wedge \theta + \Theta,$$

where Θ denotes the curvature 2-form of θ . The Ricci curvature of ∇ is the (not necessarily symmetric) covariant 2-tensor field $\mathrm{Ric}(\nabla)$ on Σ satisfying

$$\mathrm{Ric}(\nabla)(X, Y) = \mathrm{tr} \left(Z \mapsto \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z \right), \quad Z \in \Gamma(TM),$$

for all vector fields X, Y on Σ . Denoting by $\mathrm{Ric}^\pm(\nabla)$ the symmetric – and anti-symmetric part of the Ricci curvature of ∇ , so that $\mathrm{Ric}(\nabla) = \mathrm{Ric}^+(\nabla) + \mathrm{Ric}^-(\nabla)$, the (projective) Schouten tensor of ∇ is defined as

$$\mathrm{Schout}(\nabla) = \mathrm{Ric}^+(\nabla) - \frac{1}{3} \mathrm{Ric}^-(\nabla).$$

Since the components of ω are a basis for the v -semibasic forms on F ,¹ it follows that there exist real-valued functions S_{ij} on F such that

$$v^* \mathrm{Schout}(\nabla) = \omega^t S \omega = S_{ij} \omega^i \otimes \omega^j,$$

where $S = (S_{ij})$. Note that

$$R_a^* S = a^t S a$$

for all $a \in \mathrm{GL}^+(2, \mathbb{R})$, since $\omega^t S \omega$ is invariant under R_a . In terms of the functions S_{ij} the curvature 2-form $\Theta = (\Theta_j^i)$ can be written as²

$$(2.3) \quad \Theta_j^i = \left(\delta_{[k}^i S_{l]j} - \delta_j^i S_{[kl]} \right) \omega^k \wedge \omega^l,$$

¹Recall that a 1-form $\alpha \in \Omega^1(M)$ is semibasic for the projection $\pi : M \rightarrow N$ if α vanishes on vector fields that are tangent to the π -fibres.

²For a matrix $S = (S_{ij})$ we denote by $S_{(ij)}$ its symmetric part and by $S_{[ij]}$ its anti-symmetric part, so that $S_{ij} = S_{(ij)} + S_{[ij]}$.

or explicitly

$$(2.4) \quad \Theta = \begin{pmatrix} 2S_{21} - S_{12} & S_{22} \\ -S_{11} & S_{21} - 2S_{12} \end{pmatrix} \omega^1 \wedge \omega^2.$$

2.2. The orthonormal coframe bundle. Recall that if g is a Riemannian metric on the oriented surface Σ , the Levi-Civita connection (φ_j^i) of g is the unique connection on the coframe bundle $v : F \rightarrow \Sigma$ satisfying

$$\begin{aligned} d\omega^i &= -\varphi_j^i \wedge \omega^j, \\ dg_{ij} &= g_{ik}\varphi_j^k + g_{kj}\varphi_i^k, \end{aligned}$$

where we write $v^*g = g_{ij}\omega^i \otimes \omega^j$ for real-valued functions $g_{ij} = g_{ji}$ on F . Differentiating these equations implies that there exists a unique function K_g , the Gauss curvature of g , so that

$$d\varphi_j^i + \varphi_k^i \wedge \varphi_j^k = g_{jk}K_g\omega^i \wedge \omega^k.$$

We may reduce F to the $\mathrm{SO}(2)$ -subbundle F_g consisting of orientation preserving coframes that are also orthonormal with respect to g , that is, the bundle defined by the equations $g_{ij} = \delta_{ij}$. On F_g the identity $dg_{ij} = 0$ implies the identities $\varphi_1^1 = \varphi_2^2 = 0$ as well as $\varphi_2^1 + \varphi_1^2 = 0$. Therefore, writing $\varphi := \varphi_1^2$, we obtain the structure equations

$$(2.5) \quad \begin{aligned} d\omega_1 &= -\omega_2 \wedge \varphi, \\ d\omega_2 &= -\varphi \wedge \omega_1, \\ d\varphi &= -K_g\omega_1 \wedge \omega_2, \end{aligned}$$

where $\omega_i = \delta_{ij}\omega^j$. Continuing to denote the basepoint projection $F_g \rightarrow \Sigma$ by v , the area form $d\mu_g$ of g satisfies $v^*d\mu_g = \omega_1 \wedge \omega_2$. Also, note that a complex-valued 1-form α on Σ is a $(1,0)$ -form for the complex structure J induced on Σ by g and the orientation if and only if $v^*\alpha$ is a complex multiple of the complex-valued form $\omega = \omega_1 + i\omega_2$. In particular, denoting by K_Σ the canonical bundle of Σ with respect to J , a section A of the ℓ -th tensorial power of K_Σ satisfies $v^*A = a\omega^\ell$ for some unique complex-valued function a on F_g . Denote by $S_0^3(T^*\Sigma)$ the trace-free part of $S^3(T^*\Sigma)$ with respect to $[g]$. The proof of the following lemma is an elementary computation and thus omitted.

Lemma 2.1. *Suppose $S \in \Gamma(S_0^3(T^*\Sigma))$. Then there exists a unique cubic differential $C \in \Gamma(K_\Sigma^3)$ so that $\mathrm{Re}(C) = S$. Moreover, writing $v^*S = s_{ijk}\omega_i \otimes \omega_j \otimes \omega_k$ for unique real-valued functions s_{ijk} on F_g , totally symmetric in all indices, the cubic differential satisfies $v^*C = (s_{111} + is_{222})\omega^3$.*

In complex notation, the structure equations of a cubic differential $C \in \Gamma(K_\Sigma^3)$ can be written as follows. Writing $v^*C = c\omega^3$ for a complex-valued function c on F_g , it follows from the $\mathrm{SO}(2)$ -equivariance of $c\omega^3$ that there exist complex-valued functions c' and c'' on F_g such that

$$dc = c'\omega + c''\overline{\omega} + 3ic\varphi,$$

where we write $\bar{\omega} = \omega_1 - i\omega_2$. Note that the Hermitian metric induced by g on K_Σ^3 has Chern connection D given by

$$c \mapsto dc - 3ic\varphi.$$

In particular, the $(0,1)$ -derivative of C with respect to D is represented by c'' , that is, $v^*(D^{0,1}C) = c''\omega^3 \otimes \bar{\omega}$. Since $\bar{\partial} = D^{0,1}$, we obtain

$$(2.6) \quad v^*(\bar{\partial}C) = c''\omega^3 \otimes \bar{\omega}.$$

Also, we record the identity

$$v^*|C|_g^2 = |c|^2.$$

Moreover, recall that for $u \in C^\infty(\Sigma)$ we have the following standard identity for the change of the Gauss curvature of a metric g under conformal rescaling

$$K_{e^{2u}g} = e^{-2u} (K_g - \Delta_g u),$$

where $\Delta_g = -(\delta_g d + d\delta_g)$ is the negative of the Laplace–Beltrami operator with respect to g . Also,

$$d\mu_{e^{2u}g} = e^{2u} d\mu_g$$

for the change of the area form $d\mu_g$,

$$\Delta_{e^{2u}g} = e^{-2u} \Delta_g$$

for Δ_g acting on functions and

$$\delta_{e^{2u}g} = e^{-2u} \delta_g$$

for the co-differential acting on 1-forms. Finally, the norm of C changes as

$$|C|_{e^{2u}g}^2 = e^{-6u} |C|_g^2.$$

2.3. The cotangent bundle and induced structures. Recall that we have a $\mathrm{GL}^+(2, \mathbb{R})$ -representation χ on \mathbb{R}_2 – the real vector space of row vectors of length two with real entries – defined by the rule $a \cdot \xi = \xi a^{-1}$ for all $\xi \in \mathbb{R}_2$ and $a \in \mathrm{GL}^+(2, \mathbb{R})$. The cotangent bundle of Σ is the vector bundle associated to the coframe bundle F via the representation χ , that is, the bundle obtained by taking the quotient of $F \times \mathbb{R}_2$ by the $\mathrm{GL}^+(2, \mathbb{R})$ -right action induced by χ . Consequently, a 1-form on Σ is represented by an \mathbb{R}_2 -valued function ξ on F which is $\mathrm{GL}^+(2, \mathbb{R})$ -equivariant, that is, ξ satisfies $\xi(f \cdot a) = \xi(f)a$ for all $a \in \mathrm{GL}^+(2, \mathbb{R})$ and $f \in F$.

Using θ we may define a Riemannian metric h_∇ as well as a symplectic form Ω_∇ on $T^*\Sigma$ as follows. Let

$$\pi : F \times \mathbb{R}_2 \rightarrow (F \times \mathbb{R}_2) / \mathrm{GL}^+(2, \mathbb{R}) \simeq T^*\Sigma$$

denote the quotient projection. On $F \times \mathbb{R}_2$ consider the covariant 2-tensor field

$$\begin{aligned} T_\nabla &= - (d\xi + \xi\omega\xi + \omega^t S^t - \xi\theta) \omega \\ &= - \left(d\xi_i + \xi_j \omega^j \xi_i + S_{ji} \omega^j - \xi_j \theta_i^j \right) \otimes \omega^i. \end{aligned}$$

Note that the π -semibasic 1-forms on $F \times \mathbb{R}_2$ are given by the components of ω and $d\xi$. Since

$$R_a^* T_\nabla = - (d\xi a + \xi a a^{-1} \omega \xi a + \omega^t (a^{-1})^t a^t S^t a - \xi a a^{-1} \theta a) a^{-1} \omega = T_\nabla$$

for all $a \in \mathrm{GL}^+(2, \mathbb{R})$, it follows that there exists a unique symmetric covariant 2-tensor field h_∇ and a unique anti-symmetric covariant 2-tensor field Ω_∇ on $T^*\Sigma$ such that

$$\pi^* (h_\nabla + \Omega_\nabla) = T_\nabla.$$

Using the structure equation (2.1), we compute

$$\begin{aligned} \pi^* \Omega_\nabla &= - \left(d\xi_i \wedge \omega^i + \xi_i \xi_j \omega^j \wedge \omega^i + S_{ij} \omega^i \wedge \omega^j - \xi_j \theta_i^j \wedge \omega^i \right) \\ &= - (d\xi_i \wedge \omega^i + \xi_i d\omega^i + S_{ij} \omega^i \wedge \omega^j) \\ &= -d(\xi_i \omega^i) - S_{[ij]} \omega^i \wedge \omega^j. \end{aligned}$$

The 1-form $\xi\omega = \xi_i \omega^i$ on $F \times \mathbb{R}_2$ is π -semibasic and R_a invariant, hence the π -pullback of a unique 1-form τ on $T^*\Sigma$ which is the tautological 1-form of $T^*\Sigma$. Recall that the canonical symplectic form on $T^*\Sigma$ is $\Omega_0 = -d\tau$, hence Ω_∇ defines a symplectic structure on $T^*\Sigma$ which is the canonical symplectic structure twisted with the (closed) 2-form $\rho_\nabla = \frac{1}{3} \mathrm{Ric}^-(\nabla)$

$$\Omega_\nabla = \Omega_0 + \nu^* \rho_\nabla,$$

where $\nu : T^*\Sigma \rightarrow \Sigma$ denotes the basepoint projection. In particular, denoting by $o : \Sigma \rightarrow T^*\Sigma$ the zero ν -section, the definition of the Schouten tensor gives

$$o^* \Omega_\nabla = \frac{1}{3} \mathrm{Ric}^-(\nabla).$$

This shows:

Proposition 2.2. *The zero section of $T^*\Sigma$ is a Ω_∇ -Lagrangian submanifold if and only if ∇ has symmetric Ricci tensor.*

Which motivates:

Definition 2.1. A torsion-free connection ∇ on $T\Sigma$ is called *Lagrangian* if $\mathrm{Ric}^-(\nabla)$ vanishes identically.

For the symmetric part we obtain

$$\pi^* h_\nabla = - \left(d\xi_i + \xi_j \omega^j \xi_i + S_{(ij)} \omega^j - \xi_j \theta_i^j \right) \circ \omega^i = -\eta_1 \circ \omega^1 - \eta_2 \circ \omega^2$$

where we write $\eta_i = d\xi_i + \xi_j \omega^j \xi_i + S_{(ij)} \omega^j - \xi_j \theta_i^j$ and \circ denotes the symmetric tensor product. Since the four 1-forms $\eta_1, \eta_2, \omega^1, \omega^2$ are linearly independent, it follows that h_∇ is non-degenerate and hence defines a pseudo-Riemannian metric of split signature $(1, 1, -1, -1)$ on $T^*\Sigma$.

Remark 2.1. The metric h_∇ and its properties are studied in detail in [6], in particular, it is anti-self-dual and Einstein with non-zero scalar curvature.

From the definition of the Schouten tensor and h_∇ we immediately obtain

$$o^* h_\nabla = -\text{Ric}^+(\nabla),$$

which motivates:

Definition 2.2. A torsion-free connection ∇ on $T\Sigma$ is called *timelike* if $\text{Ric}^+(\nabla)$ is positive definite and *spacelike* if $\text{Ric}^+(\nabla)$ is negative definite.

3. TWISTED CONFORMAL CONNECTIONS

Let $[g]$ be a conformal structure on the smooth oriented surface Σ . By a $[g]$ -conformal connection on Σ we mean a torsion-free connection on $T\Sigma$ preserving the conformal structure $[g]$. It follows from Koszul's identity that a $[g]$ -conformal connection can be written in the following form

$$(g, \beta)\nabla = {}^g\nabla + g \otimes \beta^\sharp - \beta \otimes \text{Id} - \text{Id} \otimes \beta,$$

where $g \in [g]$, $\beta \in \Omega^1(\Sigma)$ is a 1-form and β^\sharp denotes the g -dual vector field to β . We will use the notation ${}^{[g]}\nabla$ to denote a general $[g]$ -conformal connection.

Definition 3.1. A *twisted conformal connection* ∇ on $(\Sigma, [g])$ is a torsion-free connection on the tangent bundle of Σ which can be written as $\nabla = {}^{[g]}\nabla + \alpha$ for some $[g]$ -conformal connection ${}^{[g]}\nabla$ and some 1-form α with values in $\text{End}(T\Sigma)$ satisfying the following properties:

- (i) $\alpha(X)$ is trace-free and $[g]$ -symmetric for all $X \in \Gamma(T\Sigma)$;
- (ii) $\alpha(X)Y = \alpha(Y)X$ for all $X, Y \in \Gamma(T\Sigma)$.

Note that if α satisfies the above properties, then the covariant 3-tensor obtained by lowering the upper index of α with a metric $g \in [g]$ gives a section of $\Gamma(S_0^3(T^*\Sigma))$. Conversely, every $\text{End}(T\Sigma)$ -valued 1-form on Σ satisfying the above properties arises in this way. In other words, fixing a Riemannian metric $g \in [g]$ allows to identify the twist term α with a cubic differential.

Fixing a metric $g \in [g]$, the connection form $\theta = (\theta_j^i)$ of a twisted conformal connection is given by

$$\theta_j^i = \varphi_j^i + \left(b_k g^{ki} g_{jl} - \delta_j^i b_l - \delta_l^i b_j + a_{jl}^i \right) \omega^l,$$

where the map $(g_{ij}) : F \rightarrow S^2(\mathbb{R}_2)$ represents the metric g , the map $(b_i) : F \rightarrow \mathbb{R}_2$ represents the 1-form β and the map $(a_{jk}^i) : F \rightarrow \mathbb{R}^2 \otimes S^2(\mathbb{R}_2)$ represents the 1-form α . Moreover, (φ_j^i) denote the Levi-Civita connection forms of g . Reducing to the bundle F_g of g -orthonormal orientation preserving coframes, the connection form becomes

$$\theta = \begin{pmatrix} -\beta & \varphi - \star_g \beta \\ \star_g \beta - \varphi & -\beta \end{pmatrix} + \begin{pmatrix} a_{11}^1 \omega_1 + a_{12}^1 \omega_2 & a_{12}^1 \omega^1 + a_{22}^1 \omega_2 \\ a_{11}^2 \omega_1 + a_{12}^2 \omega_2 & a_{12}^2 \omega_1 + a_{22}^2 \omega_2 \end{pmatrix},$$

where we use the identity $v^*(\star_g \beta) = -b_2 \omega_1 + b_1 \omega_2$. By definition, on F_g the functions a_{jk}^i satisfy the identities

$$a_{jk}^i = a_{kj}^i, \quad a_{kj}^k = 0, \quad \delta_{ki} a_{jl}^k = \delta_{kj} a_{il}^k.$$

Thus, writing $c_1 = a_{11}^1$ and $c_2 = a_{22}^2$, we obtain

$$\theta = \begin{pmatrix} -\beta & \star_g \beta - \varphi \\ \varphi - \star_g \beta & -\beta \end{pmatrix} + \begin{pmatrix} c_1 \omega_1 - c_2 \omega_2 & -c_2 \omega_1 - c_1 \omega_2 \\ -c_2 \omega_1 - c_1 \omega_2 & -c_1 \omega_1 + c_2 \omega_2 \end{pmatrix}.$$

In order to compute the curvature form of θ we first recall that we write $v^* \beta = b_i \omega_i$ and since $b_i \omega_i$ is $\text{SO}(2)$ -invariant, it follows that there exist unique real-valued functions b_{ij} on F_g such that

$$\begin{aligned} db_1 &= b_{11} \omega_1 + b_{12} \omega_2 + b_2 \varphi, \\ db_2 &= b_{21} \omega_1 + b_{22} \omega_2 - b_1 \varphi. \end{aligned}$$

Recall also that the area form of g satisfies $v^* d\mu_g = \omega_1 \wedge \omega_2$ and since $\star_g 1 = d\mu_g$, we get

$$v^* \delta_g \beta = -(b_{11} + b_{22}),$$

as well as

$$v^*(d\star_g \beta) = (b_{11} + b_{22})\omega_1 \wedge \omega_2.$$

Since $c_1 + ic_2$ represents a cubic differential on Σ , there exist unique real-valued functions c_{ij} on F_g such that

$$\begin{aligned} dc_1 &= c_{11} \omega_1 + c_{12} \omega_2 - 3c_2 \varphi, \\ dc_2 &= c_{21} \omega_1 + c_{22} \omega_2 + 3c_1 \varphi. \end{aligned}$$

Consequently, a straightforward calculation shows that the curvature form $\Theta = d\theta + \theta \wedge \theta$ satisfies

$$\begin{aligned} (3.1) \quad \Theta &= \begin{pmatrix} -d\beta & K_g d\mu_g + d\star_g \beta - \frac{1}{2}|\alpha|_g^2 \omega_1 \wedge \omega_2 \\ -K_g d\mu_g - d\star_g \beta + \frac{1}{2}|\alpha|_g^2 \omega_1 \wedge \omega_2 & -d\beta \end{pmatrix} \\ &+ \begin{pmatrix} 2(b_1 c_2 + b_2 c_1) - (c_{12} + c_{21}) & 2(b_1 c_1 - b_2 c_2) + (c_{22} - c_{11}) \\ 2(b_1 c_1 - b_2 c_2) + (c_{22} - c_{11}) & 2(-b_1 c_2 - b_2 c_1) + (c_{12} + c_{21}) \end{pmatrix} \omega_1 \wedge \omega_2, \end{aligned}$$

where we use the identity $v^*|\alpha|_g^2 = 4((c_1)^2 + (c_2)^2)$.

3.1. A characterisation of twisted conformal connections. We obtain a natural differential operator $D_{[g]}$ acting on the space $\mathfrak{A}(\Sigma)$ of torsion-free connections on $T\Sigma$

$$D_{[g]} : \mathfrak{A}(\Sigma) \rightarrow \Omega^2(\Sigma), \quad \nabla \mapsto \text{tr}_g \text{Ric}(\nabla) d\mu_g.$$

Note that this operator does indeed only depend on the conformal equivalence class of g . A twisted conformal connection ∇ on $(\Sigma, [g])$ can be characterised by minimising the integral of $D_{[g]}$ among its projective equivalence class $\mathfrak{p}(\nabla)$.

Proposition 3.1. *Suppose $\nabla' = [g]\nabla + \alpha$ is a twisted conformal connection on the Riemann surface $(\Sigma, [g])$. Then*

$$\inf_{\nabla \in \mathfrak{p}(\nabla')} \int_{\Sigma} D_{[g]}(\nabla) = 4\pi\chi(\Sigma) - \|\alpha\|_g^2$$

and $4\pi\chi(\Sigma) - \|\alpha\|_g^2$ is attained precisely on ∇' .

Remark 3.1. Note that

$$\|\alpha\|_g^2 = \int_{\Sigma} |\alpha|_g^2 d\mu_g$$

does only depend on the conformal equivalence class of g .

Proof of Proposition 3.1. Write $\nabla' = (g, \beta)\nabla + \alpha$ for some Riemannian metric $g \in [g]$, some 1-form β and some $\text{End}(T\Sigma)$ -valued 1-form α on Σ satisfying the properties of Definition 3.1. From (2.4) and the definition of the Schouten tensor it follows that

$$v^*(\text{tr}_g \text{Ric}(\nabla') d\mu_g) = \Theta_2^1 - \Theta_1^2,$$

where $\Theta = (\Theta_j^i)$ denotes the curvature form of ∇' pulled-back to F_g . Thus, equation (2.4) gives

$$\text{tr}_g \text{Ric}(\nabla') d\mu_g = 2K_g + 2d \star_g \beta - |\alpha|_g^2 d\mu_g$$

and hence

$$\int_{\Sigma} \text{tr}_g \text{Ric}(\nabla') d\mu_g = 4\pi\chi(\Sigma) - \|\alpha\|_g^2$$

by the Stokes and the Gauss–Bonnet theorem.

It is a classical result due to Weyl [15] that two torsion-free connections ∇^1, ∇^2 on $T\Sigma$ are projectively equivalent if and only if there exists a 1-form γ on Σ such that $\nabla^1 - \nabla^2 = \gamma \otimes \text{Id} + \text{Id} \otimes \gamma$. It follows that the connections in the projective equivalence class of ∇' can be written as

$$\nabla = \nabla' + \gamma \otimes \text{Id} + \text{Id} \otimes \gamma$$

with $\gamma \in \Omega^1(\Sigma)$. A simple computation gives

$$(3.2) \quad \text{Ric}(\nabla) = \text{Ric}(\nabla') + \gamma^2 - \text{Sym} \nabla' \gamma + 3d\gamma,$$

where $\text{Sym} : \Gamma(T^*\Sigma \otimes T^*\Sigma) \rightarrow \Gamma(S^2(T^*\Sigma))$ denotes the natural projection. We compute

$$\begin{aligned} \text{tr}_g \text{Sym} \nabla' \gamma d\mu_g &= \text{tr}_g \text{Sym} \left({}^g\nabla + g \otimes \beta^\sharp - \beta \otimes \text{Id} - \text{Id} \otimes \beta + \alpha \right) \gamma d\mu_g \\ &= d \star_g \gamma + \left(2\gamma(\beta^\sharp) - \gamma(\beta^\sharp) - \gamma(\beta^\sharp) \right) d\mu_g \\ &= d \star_g \gamma, \end{aligned}$$

where we used that $\alpha(X)$ is trace-free and $[g]$ -symmetric for all $X \in \Gamma(T\Sigma)$. Since the last summand of the right hand side of (3.2) is anti-symmetric, we

obtain

$$\begin{aligned} \int_{\Sigma} \operatorname{tr}_g \operatorname{Ric}(\nabla) d\mu_g &= \int_{\Sigma} \operatorname{tr}_g \operatorname{Ric}(\nabla) + \int_{\Sigma} \operatorname{tr}_g \gamma^2 d\mu_g - \int_{\Sigma} \operatorname{tr}_g \operatorname{Sym} \nabla' \gamma d\mu_g \\ &= 4\pi\chi(\Sigma) - \|\alpha\|_g^2 + \|\gamma\|_g^2 - \int_{\Sigma} d \star_g \gamma, \end{aligned}$$

thus the claim follows from the Stokes theorem. \square

In [12, Theorem 2.3 & Corollary 2.4] the following result is shown, albeit phrased in different language:

Proposition 3.2. *Let $(\Sigma, [g])$ be a Riemann surface. Then every torsion-free connection on $T\Sigma$ is projectively equivalent to a unique twisted $[g]$ -conformal connection.*

Let $\mathfrak{P}(\Sigma)$ denote the space of projective structures on Σ . Using Proposition 3.1 and Proposition 3.2 we immediately obtain:

Theorem 3.3. *Let $(\Sigma, [g])$ be a compact Riemann surface. Then*

$$\sup_{\mathfrak{p} \in \mathfrak{P}(\Sigma)} \inf_{\nabla \in \mathfrak{p}} \int_{\Sigma} \operatorname{tr}_g \operatorname{Ric}(\nabla) d\mu_g = 4\pi\chi(\Sigma).$$

4. MINIMAL LAGRANGIAN CONNECTIONS

4.1. The structure equations of a Lagrangian connection. We will henceforth restrict attention to torsion-free connections ∇ which are Lagrangian, so that $\operatorname{Ric}^-(\nabla) = 0$ and the Schouten tensor agrees with the Ricci tensor of ∇ . Following usual notation, we write R_{ij} instead of S_{ij} so that $\pi^* \operatorname{Ric}(\nabla) = R_{ij} \omega^i \otimes \omega^j$ with $R_{ij} = R_{ji}$. Differentiating the structure equations (2.2) implies the existence of unique real-valued functions L_i, R_{ijk} on F , of which the latter are totally symmetric in all indices, such that³

$$(4.1) \quad dR_{ij} = \left(R_{ijk} + \frac{2}{3} L_{(i} \varepsilon_{j)k} \right) \omega^k + R_{ik} \theta_j^k + R_{kj} \theta_i^k.$$

The equivariance properties of the function $R = (R_{ij})$ yield $R_a^* L = L a \det a$, where we write $L = (L_i)$. Since

$$R_a^* (\omega^1 \wedge \omega^2) = (\det a^{-1}) \omega^1 \wedge \omega^2,$$

it follows that there exists a unique 1-form $\lambda(\nabla)$ on Σ taking values in $\Lambda^2(T^*\Sigma)$, such that

$$v^* \lambda(\nabla) = (L_1 \omega^1 + L_2 \omega^2) \otimes \omega^1 \wedge \omega^2.$$

Remark 4.1. The $\Lambda^2(T^*\Sigma)$ -valued 1-form was discovered by R. Liouville and hence we call it the Liouville curvature of ∇ . It can be shown that the vanishing of $\lambda(\nabla)$ is the complete obstruction to ∇ being *projectively flat*, that is, the projective equivalence class $\mathfrak{p}(\nabla)$ defined by ∇ is flat if and only if $\lambda(\nabla)$ vanishes identically.

³We define $\varepsilon_{ij} + \varepsilon_{ji} = 0$ with $\varepsilon_{12} = 1$.

4.2. The structure equations of a minimal Lagrangian connection.

By definition, a torsion-free timelike/spacelike Lagrangian connection has the property that the zero section $o : \Sigma \rightarrow T^*\Sigma$ defines a timelike/spacelike surface in $(T^*\Sigma, h_\nabla)$ and since $\Omega_\nabla = \Omega_0$, it is also Lagrangian. In particular,

$$g = o^*h_\nabla = \pm \text{Ric}(\nabla)$$

defines a Riemannian metric on Σ . In the case where ∇ is timelike/spacelike we may ask that the image of the zero section o is a minimal surface with respect to h_∇ , that is, o has vanishing mean curvature. In this case we call ∇ itself *minimal*. Defining $\beta \in \Omega^1(\Sigma)$ by

$$\beta = \frac{3}{8} \text{tr}_g \text{Sym} \nabla g,$$

where $\text{Sym} : \Gamma(T^*\Sigma \otimes S^2(T^*\Sigma)) \rightarrow \Gamma(S^3(T^*\Sigma))$ denotes the natural projection, we have [12, Theorem 5.3]:

Theorem 4.1. *A torsion-free timelike/spacelike Lagrangian connection ∇ on $T\Sigma$ is minimal if and only if*

$$(4.2) \quad \lambda(\nabla) = \mp 2 \star_g \beta \otimes d\mu_g.$$

Example 4.2. If h is a Riemannian metric on the oriented surface Σ whose Gauss curvature K_h is positive/negative, then its Levi-Civita connection ${}^h\nabla$ is a torsion-free timelike/spacelike Lagrangian connection whose Liouville curvature is (see for instance [12])

$$\lambda({}^h\nabla) = - \star_h dK_h \otimes d\mu_h.$$

For ${}^h\nabla$ the induced metric is

$$g = \pm \text{Ric}({}^h\nabla) = \pm K_h h$$

from which one computes that $\beta = dK_h/2K_h$ and $d\mu_g = \pm K_h d\mu_h$. Hence we obtain

$$\mp 2 \star_g \beta \otimes d\mu_g = - \star_g dK_h \otimes d\mu_h = - \star_h dK_h \otimes d\mu_h = \lambda({}^h\nabla),$$

where we use the invariance of the Hodge star under conformal rescalings when acting on $\Omega^1(\Sigma)$. It follows that Levi-Civita connections of metrics of positive/negative Gauss curvature are examples of minimal Lagrangian connections. In fact, on the 2-sphere these are the only examples, see Proposition 5.2.

For what follows it is necessary to have a precise understanding of the structure equations of a torsion-free minimal Lagrangian connection ∇ . Since we assume that ∇ is timelike/spacelike, we obtain an induced metric $g = \pm \text{Ric}(\nabla)$. In particular, Σ is equipped with an integrable almost complex structure J defined by g and the orientation and we write K_Σ to denote the canonical bundle of Σ with respect to J . The minimality condition (4.2) together with (4.1) tells us that all the second and third order information of the connection ∇ is encoded in g and $\text{Sym} \nabla g$. By definition, $\text{Sym} \nabla g$ is

a section of the rank four vector bundle $S^3(T^*\Sigma)$. In the presence of the metric g we may decompose

$$S^3(T^*\Sigma) \simeq T^*\Sigma \oplus S_0^3(T^*\Sigma)$$

into a trace – and trace-free part with respect to g . Now the trace part of $\text{Sym}\nabla g$ is (up to a factor) given by the 1-form β . Lemma 2.1 implies that there exists a unique cubic differential C on Σ so that $\text{Re}(C) = \mp \frac{1}{2} \text{Sym}_0 \nabla g$, where $\text{Sym}_0 \nabla g$ denotes the trace-free part of $\text{Sym}\nabla g$ with respect to $[g]$. The structure equations can be summarised as follows:

Proposition 4.2. *Let Σ be an oriented surface and ∇ a torsion-free timelike / spacelike minimal Lagrangian connection on $T\Sigma$. Then we obtain a triple (g, β, C) on Σ consisting of a Riemannian metric $g = \pm \text{Ric}(\nabla)$, a 1-form $\beta = \frac{3}{8} \text{tr}_g \text{Sym}\nabla g$ and a cubic differential C so that $\text{Re}(C) = \mp \frac{1}{2} \text{Sym}_0 \nabla g$. Furthermore, the triple (g, β, C) satisfies the following equations*

$$(4.3) \quad K_g = \pm 1 + 2|C|_g^2 + \delta_g \beta,$$

$$(4.4) \quad \bar{\partial}C = (\beta - i \star_g \beta) \otimes C,$$

$$(4.5) \quad d\beta = 0.$$

Proof. In order to prove Proposition 4.2 we will work on the orthonormal coframe bundle F_g of g which is cut out by the equations $R_{ij} = \pm \delta_{ij}$ on F . From (4.1) we obtain

$$0 = dR_{ij} = \left(R_{ijk} + \frac{2}{3} L_{(i} \varepsilon_{j)k} \right) \omega_k \pm \delta_{ik} \theta_j^k \pm \delta_{kj} \theta_i^k.$$

Therefore, writing $\theta_{ij} = \delta_{ik} \theta_j^k$, we have

$$(4.6) \quad \theta_{(ij)} = \mp \frac{1}{2} \left(R_{ijk} + \frac{2}{3} L_{(i} \varepsilon_{j)k} \right) \omega_k.$$

Of course, the decomposition in (4.1) is so that

$$v^*(\text{Sym}\nabla g) = \pm R_{ijk} \omega_i \otimes \omega_j \otimes \omega_k,$$

hence writing $v^*\beta = b_i \omega_i$, we get

$$b_k = \pm \frac{3}{8} \delta^{ij} R_{ijk}.$$

Since

$$v^*(\star_g \beta) = b_i \varepsilon^{ij} \omega_j,$$

the minimality condition (4.2) is equivalent to

$$(L_l \omega_l) \otimes \omega_1 \wedge \omega_2 = -\frac{3}{4} \left(\delta^{ij} R_{ijk} \varepsilon^{kl} \omega_l \right) \otimes \omega_1 \wedge \omega_2,$$

or

$$L_m = -\frac{3}{4} \delta^{ij} R_{ijk} \varepsilon^{kl} \delta_{lm} = \mp 2b_k \varepsilon^{kl} \delta_{lm}.$$

From Lemma 2.1 and

$$v^*(\text{Sym}_0 \text{Ric}(\nabla)) = \left(R_{ijk} - \frac{3}{2} \delta_{(ij} R_{k)lm} \delta^{lm} \right) \omega_i \otimes \omega_j \otimes \omega_k$$

we easily compute that the cubic differential C on Σ for which $\text{Re}(C) = \mp \frac{1}{2} \text{Sym}_0 \nabla g$ satisfies

$$v^*C = \mp \left(\left(\frac{1}{8} R_{111} - \frac{3}{8} R_{122} \right) + i \left(-\frac{3}{8} R_{112} + \frac{1}{8} R_{222} \right) \right) (\omega_1 + i\omega_2)^3.$$

For later usage we introduce the notation $c_1 = \mp(\frac{1}{8} R_{111} - \frac{3}{8} R_{122})$ and $c_2 = \mp(-\frac{3}{8} R_{112} + \frac{1}{8} R_{222})$. Equation (4.6) written out gives

$$\begin{aligned} \theta_{11} &= \mp \frac{1}{2} R_{111} \omega_1 \mp \left(\frac{3}{4} R_{112} + \frac{1}{4} R_{222} \right) \omega_2, \\ \frac{1}{2}(\theta_{12} + \theta_{21}) &= \mp \left(\frac{3}{8} R_{112} - \frac{1}{8} R_{222} \right) \omega_1 \mp \left(\frac{3}{8} R_{122} - \frac{1}{8} R_{111} \right) \omega_2, \\ \theta_{22} &= \mp \left(\frac{3}{4} R_{122} + \frac{1}{4} R_{111} \right) \omega_1 \mp \frac{1}{2} R_{222} \omega_2. \end{aligned}$$

Defining

$$\varphi = \theta_{21} \mp \frac{1}{2} R_{222} \omega_1 \pm \left(\frac{1}{4} R_{111} + \frac{3}{4} R_{122} \right) \omega_2,$$

we compute

$$(4.7) \quad \theta = \begin{pmatrix} -\beta & \star_g \beta - \varphi \\ \varphi - \star_g \beta & -\beta \end{pmatrix} + \begin{pmatrix} c_1 \omega^1 - c_2 \omega^2 & -c_2 \omega^1 - c_1 \omega^2 \\ -c_2 \omega^1 - c_1 \omega^2 & -c_1 \omega^1 + c_2 \omega^2 \end{pmatrix}.$$

The motivation for the definition of φ is that we have

$$d\omega_1 = -\omega_2 \wedge \varphi \quad \text{and} \quad d\omega_2 = -\varphi \wedge \omega_1,$$

hence φ is the Levi-Civita connection form of g . In particular, we see that minimal Lagrangian connections are twisted conformal connections. Now θ is just the connection form of the spacelike minimal Lagrangian connection ∇ and since $\text{Ric}(\nabla) = \pm g$, it follows that the curvature 2-form of θ must satisfy

$$(4.8) \quad \Theta = d\theta + \theta \wedge \theta = \begin{pmatrix} 0 & \pm \omega_1 \wedge \omega_2 \\ \mp \omega_1 \wedge \omega_2 & 0 \end{pmatrix}.$$

In order to evaluate this condition we first recall that we write $v^* \beta = b_i \omega_i$ and

$$\begin{aligned} db_1 &= b_{11} \omega_1 + b_{12} \omega_2 + b_2 \varphi, \\ db_2 &= b_{21} \omega_1 + b_{22} \omega_2 - b_1 \varphi, \end{aligned}$$

for unique real-valued functions b_{ij} on F_g . From (4.7) and (4.8) we obtain

$$d\beta = -\frac{1}{2} (d\theta_{11} + d\theta_{22}) = \frac{1}{2} (\theta_{12} \wedge \theta_{21} + \theta_{21} \wedge \theta_{12}) = 0$$

showing that β is closed, hence (4.5) is verified. Likewise, we also obtain

$$\begin{aligned} d\varphi &= \frac{1}{2}(d\theta_{21} - d\theta_{12}) + d\star_g\beta \\ &= (b_{11} + b_{22})\omega_1 \wedge \omega_2 + \frac{1}{2}((\theta_{11} - \theta_{22}) \wedge (\theta_{21} + \theta_{12})) \mp \omega_1 \wedge \omega_2 \\ &= -(2((c_1)^2 + (c_2)^2) - (b_{11} + b_{22}) \pm 1)\omega_1 \wedge \omega_2. \end{aligned}$$

Writing K_g for the Gauss curvature of g , this last equation is equivalent to

$$K_g = \pm 1 + 2|C|_g^2 + \delta_g\beta,$$

which verifies (4.3).

In order to prove (4.4), we first observe

$$v^*(\beta - i\star_g\beta) = (b_1 + ib_2)(\omega^1 - i\omega^2).$$

In light of (2.6) the condition (4.4) is equivalent to the condition

$$(4.9) \quad dc \wedge \omega = b c \bar{\omega} \wedge \omega + 3ic\varphi \wedge \omega,$$

where we use the complex notation $b = b_1 + ib_2$, $c = c_1 + ic_2$ and $\omega = \omega_1 + i\omega_2$. Again, from (4.7) we compute

$$c\omega = \frac{1}{2}[(\theta_{11} - \theta_{22}) - i(\theta_{12} + \theta_{21})],$$

hence

$$\begin{aligned} dc \wedge \omega &= d(c\omega) - cd\omega = -\theta_{12} \wedge \theta_{21} \\ &\quad + \frac{i}{2}(\theta_{11} \wedge (\theta_{12} - \theta_{21}) + \theta_{22} \wedge (\theta_{21} - \theta_{12})) - (c_1 + ic_2)(d\omega_1 + id\omega_2). \end{aligned}$$

Using (4.7) and the structure equations (2.5) this gives

$$\begin{aligned} dc \wedge \omega &= 3c_2\omega_1 \wedge \varphi + 3c_1\omega_2 \wedge \varphi - 2(b_1c_2 + b_2c_1)\omega_1 \wedge \omega_2 \\ &\quad + i(-3c_1\omega_1 \wedge \varphi + 3c_2\omega_2 \wedge \varphi + 2(b_1c_1 - b_2c_2)\omega_1 \wedge \omega_2), \end{aligned}$$

which is equivalent to

$$\begin{aligned} dc \wedge \omega &= (b_1 + ib_2)(c_1 + ic_2)(\omega_1 - i\omega_2) \wedge (\omega_1 + i\omega_2) \\ &\quad + 3i(c_1 + ic_2)\varphi \wedge (\omega_1 + i\omega_2), \end{aligned}$$

that is, equation (4.9). This completes the proof. \square

Conversely, our computations also show:

Proposition 4.3. *Suppose a triple (g, β, C) on an oriented surface Σ satisfies the equations (4.3, 4.4, 4.5). Then the connection form (4.7) on F_g defines a torsion-free timelike/spacelike minimal Lagrangian connection ∇ on $T\Sigma$ with $\text{Ric}(\nabla) = \pm g$.*

We immediately obtain:

Corollary 4.4. *Let Σ be an oriented surface. Then there exists a one-to-one correspondence between torsion-free timelike/spacelike minimal Lagrangian connections on $T\Sigma$ and triples (g, β, C) satisfying (4.3, 4.4, 4.5).*

Proof. Clearly, the map sending a torsion-free minimal Lagrangian connection ∇ into the set of triples (g, β, C) satisfying the above structure equations, is surjective. Now suppose the two triples (g_1, β_1, C_1) and (g_2, β_2, C_2) on Σ satisfy the above structure equations and define the same torsion-free spacelike minimal Lagrangian connection ∇ on $T\Sigma$. Then $g_1 = \pm \text{Ric}(\nabla) = g_2$ and consequently we obtain $\beta_1 = \beta_2$ as well as $C_1 = C_2$, since these quantities are defined in terms of $\nabla \text{Ric}(\nabla)$ by using the metric $g_1 = g_2$. \square

Remark 4.3. Theorem 4.1 and Remark 4.1 immediately imply that a minimal Lagrangian connection is projectively flat if and only if β vanishes identically, or equivalently, if and only if the cubic differential C is holomorphic.

5. THE SPHERICAL CASE

The system of equations governing minimal Lagrangian connections are easy to analyse on the 2-sphere S^2 . We start with a definition.

Definition 5.1. A differential $A \in \Gamma(K_\Sigma^\ell)$ of degree ℓ on a Riemann surface Σ is called *conformally holomorphic* if locally there exists a smooth real-valued function r on Σ so that $e^{2r}A$ is holomorphic.

Conformally holomorphic differentials can be characterised as follows.

Lemma 5.1. *A differential $A \in \Gamma(K_\Sigma^\ell)$ is conformally holomorphic if and only if there exists a (1,0)-form $\lambda \in \Gamma(K_\Sigma)$ so that*

$$(5.1) \quad \bar{\partial}A - \bar{\lambda} \otimes A = 0,$$

$$(5.2) \quad \text{Re}(\bar{\partial}\lambda) = 0.$$

Proof. Suppose $A \in \Gamma(K_\Sigma^\ell)$ satisfies (5.1) for some (1,0)-form λ satisfying (5.2). We may write $\lambda = \beta + i \star \beta$ for some unique 1-form $\beta \in \Omega^1(\Sigma)$. From (5.2) we obtain

$$0 = \partial\bar{\lambda} + \bar{\partial}\lambda = \partial\bar{\lambda} + \bar{\partial}\lambda + \partial\lambda + \bar{\partial}\bar{\lambda} = (\partial + \bar{\partial})(\lambda + \bar{\lambda}) = d(\lambda + \bar{\lambda}),$$

where we have used that $d = \partial + \bar{\partial}$ and that Σ is complex one-dimensional. Consequently, the 1-form β is closed so that locally there exists a smooth real-valued function r satisfying $\lambda = -2\partial r$. Hence we obtain

$$\begin{aligned} \bar{\partial}(e^{2r}A) &= 2\bar{\partial}r \otimes e^{2r}A + e^{2r}\bar{\partial}A = -2\bar{\partial}r \otimes e^{-2r}A - e^{-2r}\bar{\lambda} \otimes A \\ &= -2\bar{\partial}r \otimes e^{-2r}A + 2\bar{\partial}r \otimes e^{-2r}A = 0, \end{aligned}$$

showing that A is conformally holomorphic. Conversely, suppose that A is conformally holomorphic and does not vanish identically, then away from the zeros of A , the (1,0)-form λ is uniquely determined by (5.1). However, since A is conformally holomorphic, the zeros of A are isolated, hence λ extends uniquely to all of Σ . \square

We now have:

Proposition 5.2. *A torsion-free connection on the tangent bundle of S^2 is minimal Lagrangian if and only if it is the Levi-Civita connection of a metric of positive Gauss curvature.*

Proof. From Lemma 5.1 we know that the cubic differential C on S^2 defined by ∇ is conformally holomorphic with respect to the complex structure induced by $g = \pm \text{Ric}(\nabla)$ and the standard orientation. Since $H^1(S^2) = 0$, the 1-form β is exact, hence C can globally be rescaled to be holomorphic. Since there are no non-trivial cubic holomorphic differentials on the 2-sphere C must vanish identically. Writing $\beta = dr$ for some smooth real-valued function r on S^2 , the connection form (4.7) of ∇ thus becomes

$$\theta = \begin{pmatrix} -dr & \star_g dr - \varphi \\ \varphi - \star_g dr & -dr \end{pmatrix},$$

where φ denotes the Levi-Civita connection form of g . Thus ∇ is a conformal connection given by

$$\nabla = {}^g\nabla + g \otimes {}^g\nabla r - dr \otimes \text{Id} - \text{Id} \otimes dr,$$

where ${}^g\nabla r$ denotes the gradient of r with respect to g . Since the Levi-Civita connection of a Riemannian metric g transforms under conformal change as [3, Theorem 1.159]

$$\exp(2f)g\nabla = {}^g\nabla - g \otimes {}^g\nabla f + df \otimes \text{Id} + \text{Id} \otimes df,$$

we obtain $\nabla = \exp(-2r)g\nabla$, thus showing that ∇ is the Levi-Civita connection of a Riemannian metric. Moreover, since $\text{Ric}(\nabla)$ must be positive or negative definite, the Gauss curvature of the metric $e^{-2r}g$ cannot vanish and hence is positive by the Gauss–Bonnet theorem. Finally, Example 4.2 shows that conversely the Levi-Civita connection of a Riemannian metric of positive Gauss curvature defines a minimal Lagrangian connection, thus completing the proof. \square

6. THE CASE OF NEGATIVE EULER-CHARACTERISTIC

In §4 we have seen that a triple (g, β, C) on an oriented surface Σ satisfying (4.3, 4.4, 4.5) uniquely determines a minimal Lagrangian connection on $T\Sigma$. In this section we will show that in the case where Σ is compact and has negative Euler characteristic $\chi(\Sigma)$, the conformal equivalence $[g]$ of g and the cubic differential C also uniquely determine (g, β, C) and hence the connection, provided C does not vanish identically. In the case where C does vanish identically the connection is determined uniquely in terms of $[g]$ and β .

We start by showing that there are no timelike minimal Lagrangian connections on a compact oriented surface of negative Euler-characteristic.

Proposition 6.1. *Suppose ∇' is a torsion-free minimal Lagrangian connection on the compact oriented surface Σ satisfying $\chi(\Sigma) < 0$. Then ∇' is spacelike.*

Proof. Suppose ∇' were timelike and let $g = \text{Ric}(\nabla')$. Then we obtain

$$\int_{\Sigma} \text{tr}_g \text{Ric}(\nabla') d\mu_g = 2 \int_{\Sigma} d\mu_g = 2 \text{Area}(\Sigma, g) \geq 0$$

and hence Proposition 3.1 and Theorem 3.3 imply that

$$4\pi\chi(\Sigma) = \sup_{\mathfrak{p} \in \mathfrak{P}(\Sigma)} \inf_{\nabla \in \mathfrak{P}} \int_{\Sigma} \text{tr}_g \text{Ric}(\nabla) d\mu_g \geq 0,$$

a contradiction. \square

6.1. Spacelike minimal Lagrangian connections. Without losing generality we henceforth assume that the torsion-free minimal Lagrangian connection ∇ on a compact oriented surface Σ with $\chi(\Sigma) < 0$ is spacelike. We will show that the triple (g, β, C) defined by ∇ is uniquely determined in terms of $[g]$ and (β, C) .

Suppose (g, β, C) with β closed satisfy

$$K_g = -1 + 2|C|_g^2 + \delta_g \beta.$$

Let g_0 denote the hyperbolic metric in $[g]$ and write $g = e^{2u} g_0$, so that

$$e^{-2u}(-1 - \Delta_{g_0} u) = -1 + 2e^{-6u}|C|_{g_0}^2 + e^{-2u}\delta_{g_0}\beta.$$

Writing $\tau = |C|_{g_0}^2 \geq 0$, we obtain

$$-\Delta_{g_0} u = 1 + \delta_{g_0}\beta - e^{2u} + 2e^{-4u}\tau.$$

Omitting henceforth reference to g_0 we will show:

Theorem 6.2. *Let (Σ, g_0) be a compact hyperbolic Riemann surface. Suppose $\beta \in \Omega^1(\Sigma)$ is closed and $\tau \in C^\infty(\Sigma)$ is non-negative. Then the equation*

$$(6.1) \quad -\Delta u = 1 + \delta\beta - e^{2u} + 2e^{-4u}\tau$$

admits a unique solution $u \in C^\infty(\Sigma)$.

Using the Hodge decomposition theorem it follows from the closedness of β that we may write $\beta = \gamma + dv$ for a real-valued function $v \in C^\infty(\Sigma)$ and a unique harmonic 1-form $\gamma \in \Omega^1(\Sigma)$. Since γ is harmonic, it is co-closed, hence (6.1) becomes

$$\Delta u = -1 - \delta dv + e^{2u} - 2e^{-4u}\tau = -1 + \Delta v + e^{2u} - 2e^{-4u}\tau.$$

Writing $u' := u - v$, we obtain

$$\Delta u' = -1 + e^{2(u'+v)} - 2e^{-4(u'+v)}\tau.$$

Using the notation $\kappa = -e^{2v} < 0$ and $\xi = \tau e^{-4v}$, as well as renaming $u := u'$, we see that Theorem (6.2) follows from:

Theorem 6.3. *Let (Σ, g_0) be a compact hyperbolic Riemann surface. Suppose $\kappa, \xi \in C^\infty(\Sigma)$ satisfy $\kappa < 0$ and $\xi \geq 0$. Then the equation*

$$(6.2) \quad -\Delta u = 1 + \kappa e^{2u} + 2\xi e^{-4u}$$

admits a unique solution $u \in C^\infty(\Sigma)$.

In order to prove this theorem we use the direct method in the calculus of variations for an appropriate functional $\mathcal{E}_{\kappa, \xi}$ defined on the Sobolev space $W^{1,2}(\Sigma)$. As usual, we say a function $u \in W^{1,2}(\Sigma)$ is a *weak solution* of (6.2) if for all $\phi \in C^\infty(\Sigma)$

$$(6.3) \quad 0 = \int_{\Sigma} -\langle du, d\phi \rangle + (1 + \kappa e^{2u} + 2\xi e^{-4u}) \phi d\mu.$$

Note that this definition makes sense. Indeed, it follows from the Moser–Trudinger inequality that the exponential map sends the Sobolev space $W^{1,2}(\Sigma)$ into $L^p(\Sigma)$ for every $p < \infty$, hence the right hand side of (6.3) is well defined.

Lemma 6.4. *Suppose $u \in W^{1,2}(\Sigma)$ is a critical point of the functional*

$$\mathcal{E}_{\kappa, \xi} : W^{1,2}(\Sigma) \rightarrow \mathbb{R}, \quad u \mapsto \frac{1}{2} \int_{\Sigma} |du|^2 - 2u - \kappa e^{2u} + \xi e^{-4u} d\mu.$$

Then $u \in C^\infty(\Sigma)$ and u solves (6.2).

Proof. For $u, v \in W^{1,2}(\Sigma)$ we define $\gamma_{u,v}(t) = u + tv$ for $t \in \mathbb{R}$. We consider the curve $\Gamma_{u,v} = \mathcal{E}_{\kappa, \xi} \circ \gamma_{u,v} : \mathbb{R} \rightarrow \mathbb{R}$ so that

$$(6.4) \quad \Gamma_{u,v}(t) = \frac{1}{2} \int_{\Sigma} |du|^2 + 2t \langle du, dv \rangle + t^2 |dv|^2 \\ - 2(u + tv) - \kappa e^{2(u+tv)} + \xi e^{-4(u+tv)} d\mu.$$

The curve $\Gamma_{u,v}(t)$ is differentiable in t with derivative

$$\frac{d}{dt} \Gamma_{u,v}(t) = \int_{\Sigma} \langle du, dv \rangle + t |dv|^2 - v - v \kappa e^{2(u+tv)} - 2v \xi e^{-4(u+tv)} d\mu.$$

Note that this last expression is well-defined. Again, it follows from the Moser–Trudinger inequality that $e^{2(u+tv)} \in L^2(\Sigma)$ for all $u, v \in W^{1,2}(\Sigma)$ and $t \in \mathbb{R}$. Since $W^{1,2}(\Sigma) \subset L^2(\Sigma)$ it follows that $ve^{2(u+tv)}$ is in $L^1(\Sigma)$ by Hölder’s inequality and thus so is $ve^{-4(u+tv)}$. In particular, assuming that u is a critical point and setting $t = 0$ after differentiation gives

$$0 = \left. \frac{d}{dt} \right|_{t=0} \Gamma_{u,v}(t) = \int_{\Sigma} \langle du, dv \rangle - v - v \kappa e^{2u} - 2v \xi e^{-4u} d\mu.$$

Since $C^\infty(\Sigma) \subset W^{1,2}(\Sigma)$ it follows that u is a weak solution of (6.2). Since the right hand side of (6.2) is in $L^p(\Sigma)$ for all $p < \infty$, it follows from the Calderón–Zygmund inequality that $u \in W^{2,p}(\Sigma)$ for any $p < \infty$. Therefore, by the Sobolev embedding theorem, u is an element of the Hölder space $C^{1,\alpha}(\Sigma)$ for any $\alpha < 1$. Since the right hand side of (6.2) is Hölder continuous

in u , it follows from Schauder theory that $u \in C^2(\Sigma)$, so that u is a classical solution of (6.2). Iteration of the Schauder estimates then gives that $u \in C^\infty(\Sigma)$. \square

Since $\xi \geq 0$ we have $\mathcal{E}_{\kappa,\xi} \geq \mathcal{E}_{\kappa,0}$ where here 0 stands for the zero-function. The functional $\mathcal{E}_{\kappa,0}$ appears in the variational formulation of the equation for prescribed Gauss curvature κ of a metric $g = e^{2u}g_0$ on Σ . In particular, $\mathcal{E}_{\kappa,0}$ is well-known to be coercive and hence so is $\mathcal{E}_{\kappa,\xi}$. In addition, we have:

Lemma 6.5. *The functional $\mathcal{E}_{\kappa,\tau}$ is strictly convex on $W^{1,2}(\Sigma)$.*

Proof. Let $u, v \in W^{1,2}(\Sigma)$ be given. Using the notation of the previous lemma, we observe that $\Gamma_{u,v}(t)$ is twice differentiable in t with derivative

$$(6.5) \quad \frac{d^2}{dt^2} \Gamma_{u,v}(t) = \int_{\Sigma} |dv|^2 - 2v^2 \kappa e^{2(u+tv)} + 8v^2 \xi e^{-4(u+tv)} d\mu.$$

Note again that by Sobolev embedding $v^2 \in L^2(\Sigma)$ for $v \in W^{1,2}(\Sigma)$ and that both $e^{2(u+tv)}$ and $e^{-4(u+tv)}$ are in $L^2(\Sigma)$, hence the right hand side of the equation (6.5) is well-defined by Hölder's inequality. In particular, computing the second variation gives

$$\begin{aligned} \mathcal{E}_{\kappa,\tau}''(u)[v, v] &= \left. \frac{d^2}{dt^2} \right|_{t=0} \mathcal{E}_{\kappa,\tau}(u + tv) \\ &= \int_{\Sigma} |dv|^2 d\mu + 2 \int_{\Sigma} v^2 (4\xi - e^{6u} \kappa) e^{-4u} d\mu \\ &\geq \|dv\|_{L^2(\Sigma)}^2, \end{aligned}$$

where we have used that $\xi \geq 0$ and $\kappa < 0$. Since for a non-zero constant function v we obviously have $\mathcal{E}_{\kappa,\tau}''(u)[v, v] > 0$ it follows that the quadratic form $\mathcal{E}_{\kappa,\tau}''$ is positive definite on $W^{1,2}(\Sigma)$. Hence, the claim is proved. \square

Proof of Theorem 6.3. We have shown that $\mathcal{E}_{\kappa,\xi}$ is a continuous strictly convex coercive functional on the reflexive Banach space $W^{1,2}(\Sigma)$, hence $\mathcal{E}_{\kappa,\xi}$ attains a unique minimum on $W^{1,2}(\Sigma)$, see for instance [13]. Since we know that the minimum is smooth, Theorem 6.3 is proved. \square

We have seen that the Gauss curvature of the metric $g = -\text{Ric}(\nabla)$ defined by a spacelike minimal Lagrangian connection ∇ satisfies

$$K_g = -1 + 2|C|_g^2 + \delta_g \beta.$$

Integrating against $d\mu_g$ and using the Stokes and Gauss–Bonnet theorem gives

$$2\pi\chi(\Sigma) = -\text{Area}(\Sigma, g) + 2\|C\|_g^2,$$

so that we obtain the area bound

$$\text{Area}(\Sigma, g) \geq -2\pi\chi(\Sigma).$$

Definition 6.1. We call a spacelike minimal Lagrangian connection ∇ *area minimising* if (Σ, g) has area $-2\pi\chi(\Sigma)$.

Clearly, a spacelike minimal Lagrangian connection is area minimising if and only if the induced cubic differential vanishes identically.

Theorem 6.2 shows that the triple (g, β, C) is uniquely determined in terms of the conformal equivalence class $[g]$, the cubic differential C and the 1-form β on Σ . Since C can locally be rescaled to be holomorphic, its zeros must be isolated and hence β is uniquely determined by C provided C does not vanish identically. Therefore, applying Corollary 4.4 shows:

Theorem 6.6. *Let Σ be a compact oriented surface with $\chi(\Sigma) < 0$. Then we have:*

- (i) *there exists a one-to-one correspondence between area minimising Lagrangian connections on $T\Sigma$ and pairs $([g], \beta)$ consisting of a conformal structure $[g]$ and a closed 1-form β on Σ ;*
- (ii) *there exists a one-to-one correspondence between minimal Lagrangian connections on $T\Sigma$ (that are not area minimising) and pairs $([g], C)$ consisting of a conformal structure $[g]$ and a cubic differential C on Σ (which does not vanish identically) and which is conformally holomorphic with respect to the conformal structure induced by $[g]$ and the orientation.*

REFERENCES

- [1] D. BARAGLIA, G_2 geometry and integrable systems, PhD thesis, 2010. [arXiv 1002.1767](#). [2](#)
- [2] Y. BENOIST and D. HULIN, Cubic differentials and finite volume convex projective surfaces, *Geom. Topol.* **17** (2013), 595–620. [MR 3039771](#) [Zbl 1266.30030](#) [1](#)
- [3] A. L. BESSE, *Einstein manifolds*, *Ergebnisse der Mathematik und ihrer Grenzgebiete* (3) **10**, Springer-Verlag, Berlin, 1987. [MR 867684](#) [Zbl 0613.53001](#) [17](#)
- [4] S. CHOI and W. M. GOLDMAN, Convex real projective structures on closed surfaces are closed, *Proc. Amer. Math. Soc.* **118** (1993), 657–661. [MR 1145415](#) [Zbl 0810.57005](#) [1](#)
- [5] D. DUMAS and M. WOLF, Polynomial cubic differentials and convex polygons in the projective plane, *Geom. Funct. Anal.* **25** (2015), 1734–1798. [MR 3432157](#) [Zbl 06526259](#) [1](#)
- [6] M. DUNAJSKI and T. METTLER, Gauge theory on projective surfaces and anti-self-dual Einstein metrics in dimension four, 2015. [arXiv 1509.04276](#). [1](#), [2](#), [7](#)
- [7] N. HITCHIN, The self-duality equations on a Riemann surface, *Proc. London Math. Soc.* (3) **55** (1987), 59–126. [MR 887284](#) [Zbl 0634.53045](#) [2](#)
- [8] ———, Lie groups and Teichmüller space, *Topology* **31** (1992), 449–473. [MR 1174252](#) [Zbl 0769.32008](#) [1](#)
- [9] F. LABOURIE, Flat projective structures on surfaces and cubic holomorphic differentials, *Pure Appl. Math. Q.* **3** (2007), 1057–1099. [MR 2402597](#) [Zbl 1158.32006](#) [1](#), [2](#)
- [10] J. LOFTIN, Survey on affine spheres, in *Handbook of geometric analysis, No. 2, Adv. Lect. Math. (ALM)* **13**, Int. Press, Somerville, MA, 2010, pp. 161–191. [MR 2743442](#) [Zbl 1214.53013](#) [2](#)
- [11] J. C. LOFTIN, Affine spheres and convex \mathbb{RP}^n -manifolds, *Amer. J. Math.* **123** (2001), 255–274. [MR 1828223](#) [Zbl 0997.53010](#) [1](#)
- [12] T. METTLER, Extremal conformal structures on projective surfaces, 2015. [arXiv 1510.01043](#). [2](#), [11](#), [12](#)

- [13] M. STRUWE, *Variational methods*, fourth ed., *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)* **34**, Springer-Verlag, Berlin, 2008. [MR 2431434](#) [Zbl 1284.49004](#) [20](#)
- [14] C. P. WANG, Some examples of complete hyperbolic affine 2-spheres in \mathbf{R}^3 , in *Global differential geometry and global analysis (Berlin, 1990)*, *Lecture Notes in Math.* **1481**, Springer, Berlin, 1991, pp. 271–280. [MR 1178538](#) [Zbl 0743.53004](#) [2](#)
- [15] H. WEYL, Zur Infinitesimalgeometrie: Einordnung der projektiven und der konformen Auffassung., *Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl.* **1921** (1921), 99–112. [Zbl 48.0844.04](#) [10](#)

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